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VIBRATION FREQUENCIES OF A MEMBRANE STRIP WITH WAVY BOUNDARIES

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1. INTRODUCTION AND FORMULATION

The study of the vibration of membranes is important in the design of drums, receivers and loudspeakers. Membrane shapes in the form of rectangles, circles, circular sectors, and ellipses have been studied [1, 2]. The case for a perturbed circular boundary was discussed in reference [1], where it was found that the frequency is always increased. The purpose of the present note is to investigate whether a phase difference in the boundary waviness has any effect. A membrane strip with wavy boundaries shall be considered, described in Cartesian co-ordinates (x', y') by

$$y' = a \sin(\lambda x'/L), \qquad y' = L + a \sin(\lambda x'/L + \beta), \tag{1}$$

where L is the mean width, a is the amplitude, $2\pi L/\lambda$ is the period of corrugations, and β is a phase difference (Figure 1). One normalizes all lengths by L, drop primes, and assume $\varepsilon = a/L \ll 1$. The boundaries become

$$y = \varepsilon \sin(\lambda x), \qquad y = 1 + \varepsilon \sin(\lambda x + \beta). \tag{2}$$

The governing equation is

$$w_{xx} + w_{yy} + k^2 w = 0, \tag{3}$$

where w is the vertical displacement and k is the normalized vibrational frequency, i.e., (frequency) $L/\sqrt{[(\text{tension per length})/(\text{mass per area})]}$. One seeks the eigenvalue k when w is zero on the boundaries.

2. PERTURBATION SOLUTION

If ε were zero, the solution for the gravest (most important) mode is $w = \sin ky$ where $k = \pi$. The effect of waviness on this mode is now investigated. Let

$$k = \pi(1 + \varepsilon^2 b + 0(\varepsilon^4)). \tag{4}$$

The expansion in ε^2 is due to the fact that frequency is an even function of amplitude ε. The constant b is to be determined. One also expands w:

$$w = w_0(y) + \varepsilon w_1(x, y) + \varepsilon^2 w_2(x, y) + 0(\varepsilon^3), \tag{5}$$

where

$$w_0 = \sin \pi y. \tag{6}$$

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The boundary condition that w is zero at $y = \varepsilon \sin(\lambda x)$ gives

$$\begin{aligned} 0 = w|_{\varepsilon \sin \lambda x} &= w|_0 + \varepsilon \sin(\lambda x) w_y|_0 + (\varepsilon^2 \sin^2(\lambda x)/2) w_{yy}|_0 + \cdots \\ &= w_0|_0 + \varepsilon(w_1|_0 + \sin(\lambda x) w_{0y}|_0) \\ &\quad + \varepsilon^2 (w_2|_0 + \sin(\lambda x) w_{1y}|_0 + \tfrac{1}{2}\sin^2(\lambda x) w_{0yy}|_0) + \cdots . \end{aligned} \tag{7}$$

Similarly, on the other side,

$$\begin{aligned} 0 = w|_{1+\varepsilon \sin(\lambda x+\beta)} &= w_0|_1 + \varepsilon(w_1|_1 + \sin(\lambda x + \beta) w_{0y}|_1 \\ &\quad + \varepsilon^2(w_2|_1 + \sin(\lambda x+\beta) w_{1y}|_1 + \tfrac{1}{2}\sin^2(\lambda x + \beta) w_{0yy}|_1) + \cdots . \end{aligned} \tag{8}$$

Equations (3–5) give the first order correction

$$w_{1xx} + w_{1yy} + \pi^2 w_1 = 0. \tag{9}$$

Equations (7, 8) give the boundary conditions

$$w_1(x, 0) = -\sin(\lambda x) w_{0y}|_0 = -\pi \sin(\lambda x), \tag{10}$$

$$w_1(x, 1) = -\sin(\lambda x + \beta) w_{0y}|_1 = \pi \sin(\lambda x + \beta). \tag{11}$$

Let

$$w_1(x, y) = e^{i\lambda x}\phi(y), \tag{12}$$

where only the real part of the product has any physical significance. Then equations (9–11) become

$$\phi''(y) + (\pi^2 - \lambda^2)\phi = 0, \qquad \phi(0) = i\pi, \quad \phi(1) = -i\pi\, e^{i\beta}. \tag{13, 14}$$

The solution is

$$\phi y = \begin{cases} i\pi \cos(\sqrt{\pi^2-\lambda^2}\,y) - i\pi(e^{i\beta} + \cos\sqrt{\pi^2-\lambda^2})\dfrac{\sin(\sqrt{\pi^2-\lambda^2}\,y)}{\sin\sqrt{\pi^2-\lambda^2}}, & \text{if } \lambda < \pi. \\ i\pi - i\pi(1+e^{i\beta})y, & \text{if } \lambda = \pi, \\ i\pi \cosh(\sqrt{\lambda^2-\pi^2}\,y) - i\pi(e^{i\beta} + \cosh\sqrt{\lambda^2-\pi^2})\dfrac{\sinh(\sqrt{\lambda^2-\pi^2}\,y)}{\sinh\sqrt{\lambda^2-\pi^2}}, & \text{if } \lambda > \pi. \end{cases} \tag{15}$$

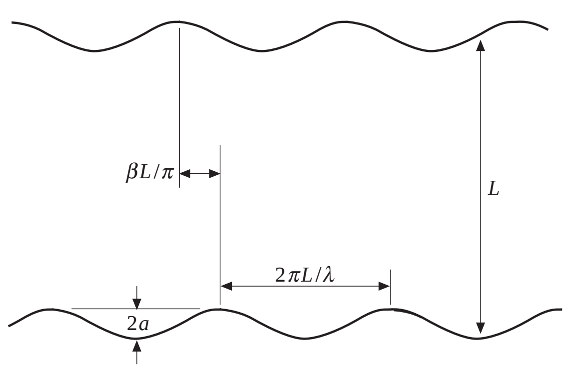

Figure 1. The geometry of the membrane strip.

The next order problem is more involved:

$$w_{2xx} + w_{2yy} + \pi^2 w_2 = -2\pi^2 b w_0 = -2\pi^2 b \sin(\pi y), \tag{16}$$

$$w_2(x, 0) = -\sin(\lambda x)\, e^{i\lambda x} \phi'(0) = (i/2)\,(e^{2i\lambda x} - 1)\phi'(0), \tag{17}$$

$$w_2(x, 1) = -\sin(\lambda x + \beta)\, e^{i\lambda x} \phi'(1) = \frac{i}{2}\,(e^{2i\lambda x + i\beta} - e^{-i\beta})\phi'(1). \tag{18}$$

Here, the full complex form of $\sin(\lambda x)$ and $\sin(\lambda x + \beta)$ has been carefully inserted. Guided by the boundary conditions set,

$$w_2(x, y) = \psi_0(y) + \psi_2(y)\, e^{2i\lambda x}. \tag{19}$$

Then,

$$\psi_0''(y) + \pi^2 \psi_0 = -2\pi^2 b \sin(\pi y), \tag{20}$$

$$\psi_0'(0) = (-i/2)\phi'(0), \qquad \psi_0(1) = (-i/2)\, e^{-i\beta} \phi'(1), \tag{21}$$

$$\psi_2''(y) + (\pi^2 - 4\lambda^2)\psi_2 = 0, \tag{22}$$

$$\psi_2(0) = (i/2)\phi'(0), \qquad \psi_2(1) = (i/2)\, e^{i\beta} \phi'(1). \tag{23}$$

The solution of ψ_2 will not be investigated, which has three forms, depending on whether 2λ is greater, equal or less than π. More interesting is the solution of ψ_0 which gives the frequency changes.

From equation (20), the general solution is

$$\psi_0 = c_1 \sin(\pi y) + c_2 \cos(\pi y) + b\pi y \cos(\pi y). \tag{24}$$

The first term $\sin(\pi y)$ can be absorbed into w_0, or one can set $c_1 = 0$ without loss of generality. Using the boundary conditions (21) one finds $c_2 = -i\phi'(0)/2$ and after some work,

$$b = \frac{i}{2\pi}\,[\phi'(0) + e^{-i\beta}\phi'(1)] = \begin{cases} \dfrac{\sqrt{\pi^2 - \lambda^2}}{\sin\sqrt{\pi^2 - \lambda^2}}\,[\cos\sqrt{\pi^2 - \lambda^2} + \cos\beta], & \lambda < \pi, \\ 1 + \cos\beta, & \lambda = \pi, \\ \dfrac{\sqrt{\lambda^2 - \pi^2}}{\sinh\sqrt{\lambda^2 - \pi^2}}\,[\cosh\sqrt{\lambda^2 - \pi^2} + \cos\beta], & \lambda > \pi. \end{cases} \tag{25}$$

3. RESULTS AND DISCUSSION

The relative frequency change is proportional to amplitude a squared and the parameter b. For given phase difference β, b is plotted against period λ in Figure 2. In general b increases with λ and approaches $\sqrt{\lambda^2 - \pi^2}$ for large λ. For small λ, $b \sim -2\pi^2(1 - \cos\beta)/\lambda^2$ which is large and negative for $\beta \neq 0$. Thus for in-phase boundaries ($\beta = 0$) the frequency of vibration is always increased, while the frequency may be increased or decreased if the waviness is out of phase, with the maximum decrease at 180° out of phase. This result differs from that of a perturbed circle, where Rayleigh always found an increase in pitch. From equation (25) one finds that the frequency is unchanged ($b = 0$) if phase and period are such that

$$\lambda = \sqrt{\beta(2\pi - \beta)}. \tag{26}$$

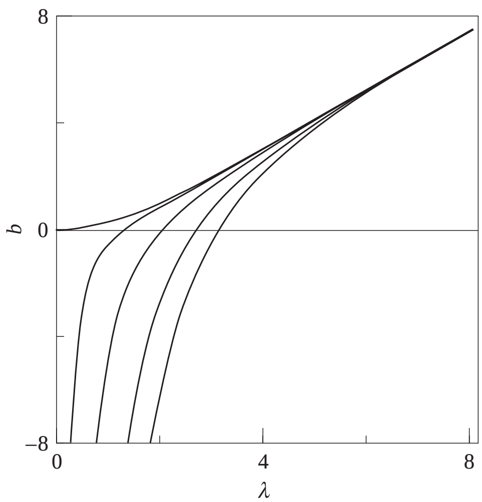

Figure 2. The frequency parameter b as a function of wave number λ for various phase difference β. From top to bottom: $\beta = 0$, $0{\cdot}1\pi$, $0{\cdot}25\pi$, $0{\cdot}5\pi$, π.

This relation is also reflected in Figure 2. Since fastening a membrane at discrete points on the boundary mimics a perturbed wavy boundary, equation (26) implies there is an optimum spacing and phase shift for the fastening points such that changes in frequency may be minimized.

REFERENCES

1. LORD RAYLEIGH 1945 *The Theory of Sound*, volume I, second edition. New York: Dover; Chapter 9.
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